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A matrix model dual of type 0B string theory in two dimensions

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Abstract: We propose that type 0B string theory in two dimensions admits a dual description in terms of a one dimensional bosonic matrix model of a hermitean matrix. The potential in the matrix model is symmetric with respect to the parity-like Z_2 transformation of the matrix. The two sectors in the theory, namely the NSNS and RR scalar sectors correspond to two classes of operators in the matrix model, even and odd under the Z_2 symmetry respectively. We provide evidence that the matrix model successfully reconstructs the perturbative S-matrix of the string theory, and reproduces the closed string emission amplitude from unstable D-branes. Following recent work in two dimensional bosonic string, we argue that the matrix model can be identified with the theory describing N unstable D0-branes in type 0B theory. We also argue that type 0A theory is described in terms of the quantum mechanics of brane-antibrane systems.

Keywords: Superstrings and Heterotic Strings, D-branes, Tachyon Condensation, 2D Gravity.

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Contents

1. Introduction

As is well known, two dimensional bosonic string theory admits a nonperturbative description in terms of a one dimensional matrix model, the $c = 1$ matrix model (for a comprehensive review of this duality see [1]–[3]). The matrix model describes the quantum mechanics of an $N \times N$ hermitean matrix Φ in some potential. On-shell closed string tachyon vertex operators correspond to the 'puncture' operator in the matrix model as follows

$$e^{iP(X_0+\phi)}e^{2\phi} \;\leftrightarrow\; \lim_{l\to 0}\int dx e^{iPx}\,\mathrm{Tr}\, e^{-l\Phi(x)}\,. \tag{1.1}$$

Recently, a clearer understanding of this mysterious duality has emerged. As argued in references [4, 5, 6], the $N \times N$ hermitean matrix Φ should be identified with the open string tachyon field T on N unstable D0-branes in the bosonic string theory. Therefore, this duality is another example of holographic open-closed string dualities. It fits nicely in the web of holographic dualities such as Matrix theory [7] and the AdS/CFT correspondence [8].

In this paper we propose a similar description of the fermionic string in two dimensions in terms of a one dimensional matrix model. In particular type 0B string theory in two dimensions is described in terms of the quantum mechanics of a hermitean $N \times N$ matrix T with a potential symmetric under $T \leftrightarrow -T$: namely $U(T) = U(-T)$. The matrix model we have in mind is the model describing a collection of N unstable D0-branes in this theory.[1] The hermitean matrix T is identified with the open string tachyon field on the D0-branes. Thus we obtain another example of open-closed string duality.

[1] In flat ten dimensions, type 0 theory has doubled spectrum of D-branes [9]–[11]. However, as shown in [12, 13], there is only one type of spin structure allowed for boundary states in $N = 1$ super Liouville theory. Thus in the two dimensional fermionic string, we only have one type of unstable D0-branes.

The Z_2 symmetry of the potential in the fermionic case arises due to the fact that the open string tachyon potential in the theory of D0-branes obeys such symmetry. This symmetry is crucial in our analysis. The form of the potential is such that a maximum at $T = 0$ separates two stable minima on each side. Unlike the bosonic case where the potential is believed to be unbounded from below having only a metastable minimum, the vacua in the fermionic case are stable. Both sides of the potential can be filled with fermion eigenvalues resulting in a fermi sea that is left/right symmetric. We propose that such a symmetric ground state corresponds to the 0B vacuum. Thus unlike the case of the bosonic string, where only one side of the potential is filled, the closed string vacuum in the fermionic case is stable even non–perturbatively. In the bosonic case, the vacuum is non-perturbatively unstable since tunneling effects can lead to particle loss on the unbounded side (for recent discussions on the relation between non-perturbative effects [14] in the matrix model and D-branes see [15, 16]).

Operators in the matrix model fall naturally into two sectors. Those operators which are even under the Z_2 symmetry and those which are odd. These operators describe disturbances of the fermi sea which are left and right symmetric and antisymmetric respectively. We propose to identify each sector of operators of definite parity with the two sectors of type 0B theory. The NSNS scalar admits a dual description in terms of even operators in the matrix model while the RR scalar in terms of odd operators. We provide evidence for such a correspondence by comparing scattering amplitudes on the super Liouville side with matrix model computations. We provide another check of the correspondence by computing the closed string emission amplitude [5, 6, 17, 18] during the process of open string tachyon condensation [19]–[28]. As we will see, this gives a precise relation between operators in the matrix model and closed string states in the NSNS and RR sectors.

In the more familiar example of the AdS/CFT correspondence, a similar relation to eq. (1.1) occurs between on-shell closed string states and gauge theory operators. Expectation values of closed string bulk fields appear as couplings of the gauge theory operators. A similar picture seems to emerge in the matrix model description of two dimensional string theory. Indeed operators of the form (1.1) in the theory of unstable D0-branes couple naturally to the closed string tachyon.

The dependence on the tachyon field in the exponential in (1.1) can be understood intuitively as follows. Such operators in the matrix model cut loops of proper length l in the dual Riemann surface [1, 29, 30]. In the presence of an open string tachyon, a boundary interaction is added to the worldsheet conformal field theory given by $\int_{\partial\Sigma} d\sigma T \sim lT$. The exponential form of this interaction appears as in boundary string field theory (BSFT) [31]–[33].

Motivated by such observations, we propose in the fermionic case (type 0B theory) the following operator as 'dual' to the tachyon field in the NSNS sector

$$\int dx e^{iPx}\, \mathrm{Tr}\ e^{-lT^2(x)}\,. \tag{1.2}$$

The gaussian form is implied by world-sheet supersymmetry as is familiar in BSFT [34, 35].

For the RR sector, we propose the odd function analogue

$$\int dx e^{iPx} \operatorname{Tr}\ T(x) e^{-lT^2(x)} \tag{1.3}$$

of (1.2). These operators yield the correct leg factors that appear in the scattering amplitudes of super Liouville theory.

The paper is organized as follows. In section 2, we review some useful results in super Liouville theory. We describe how to obtain the type 0A and type 0B theories in two dimensions and their spectra. We summarize the results for closed string scattering amplitudes in these theories obtained in [36]. We also study the decay of unstable D0-branes in 0B theory. In section 3, we propose a matrix model description for 0B theory in two dimensions. We explain how to obtain the string scattering amplitudes from correlation functions in the matrix model with the correct leg factors. We further illuminate the correspondence by identifying decaying branes in string theory with rolling eigenvalues in the matrix model and macroscopic loop operators in the matrix model with Euclidean D-branes. We also mention the matrix model dual of type 0A theory. In section 4 we include a brief discussion.

As we were finishing this work, we learned of related work to appear by M. Douglas, I. Klebanov, D. Kutasov, J. Maldacena, E. Martinec and N. Seiberg [37]. I. Klebanov has presented similar ideas to ours in his talk at strings 2003 [38].

2. $N = 1$ non-critical closed string

The $N = 1$ super Liouville theory is defined by the following action

$$S = \frac{1}{2\pi} \int dz^2 \int d\theta^2 [D\Phi \bar{D}\Phi + 2\mu_0 e^{b\Phi}] , \tag{2.1}$$

where Φ is a superfield. The central charge is given by $c = \frac{3}{2}(1 + 2Q^2)$, where $Q = b + 1/b$. The conformal dimension of the primary operators $V_\alpha = e^{\alpha\phi}$ is given by $\alpha(Q - \alpha)/2$. On the string theory side we always set $\alpha' = 2$.

The $N = 1$ two dimensional string (or fermionic $\hat{c} = 1$ non-critical string theory) [39, 40, 36] is obtained by setting $b = 1, Q = 2$ and adding a world-sheet scalar field Y together with its super partner ψ. The total central charge is canceled by b, c, β, γ ghosts ($c_{gh} = -15$). We will implicitly consider the analytical continuation $Y = iX^0$ into two dimensional Minkowski space. In this meaning, we also use the vector notation $X^\mu = (X^0, \phi)$.

Since the Liouville term in the action is obtained by adding the tachyon vertex operator, we consider type 0 theory (non-chiral GSO projection) in order not to project it out. The field content in NSNS sector consists only of the 'tachyon field' T_{cl}, which becomes massless due to Liouville dressing (for a review of quantum Lioville theory see [41, 42]). Its vertex operator in the (-1) picture is

$$V_{NSNS} = e^{-\varphi} e^{iPX_0} e^{(\frac{Q}{2} \pm iP)\phi} , \tag{2.2}$$

where only one linear combination (of $\pm$ sign) is allowed for the wave function due to reflection at the Liouville wall; thus we can choose the $+$ sign [42].

In the RR-sector, we have two choices of GSO projection: type 0A and 0B. This fact is shown by requiring the locality of OPEs as in the ten dimensional type 0A and 0B theories (for a review of ten dimensional type 0 string theory see [43]). To be more explicit, let us employ the bosonization of fermions $\psi_\phi + i\psi = e^{ih}$ (h is the new boson). Then the physical R vertex in the $-\frac{1}{2}$ picture is given by

$$V_{R(\epsilon)} = e^{-\frac{1}{2}\varphi} e^{\frac{i}{2}\epsilon h} e^{iPX_0 + (\frac{Q}{2} - i\epsilon P)\phi} \,. \tag{2.3}$$

Note that for fixed ϵ the operator has only a left or a right-moving mode. The total RR vertex can be constructed by left and right combination of the above operators $V_{R(\epsilon_L)} \otimes V_{R(\epsilon_R)}$. The type 0A theory corresponds to the choice $(\epsilon_L, \epsilon_R) = (+,-), (-,+)$, while type 0B to $(\epsilon_L, \epsilon_R) = (+,+), (-,-)$.

Since the RR sector of type 0B theory is left-right symmetric, it has a well-defined RR-scalar field C. Its field strength $F_\mu = \partial_\mu C$ is divided into two (light-cone coordinate) components[2] F_+ and F_-. Indeed F_+ corresponds to $(\epsilon_L, \epsilon_R) = (+,+)$ and F_- to $(-,-)$. Note also that this RR-form is middle dimensional as RR-5 form in ten dimensional type 0B string and that there is no self-dual constraint in type 0 theory. In this way type 0B theory includes two massless scalar fields T_{cl} and C.

On the other hand, in type 0A theory the RR-vertex operator is like $\sim e^{iP(X_L - X_R)}$ and makes sense only at zero momentum (or constant RR-flux). This is understood as follows. In two dimensional type 0A theory it is natural to expect RR-1 form potential C'. Its 2-form field strength $F' = dC'$ should obey the equation of motion $d * F' = 0$ and this allows only constant flux (much like the massive IIA theory).

2.1 Closed string scattering amplitudes and leg factors

The three-point and four-point closed string scattering amplitudes were computed at tree level in [36, 44]. In the NSNS sector the result is given by[3] (where we speculate the results for higher point functions very naturally)

$$A(P_1, P_2, \ldots, P_m) = \mu^{2-m} \left[\prod_{j=1}^{m} \mu^{-iP_i} \gamma(1 + iP_j) \right] S(\sqrt{2}P_1, \sqrt{2}P_2, \ldots, \sqrt{2}P_m) \,, \tag{2.4}$$

where $\gamma(x) = \Gamma(x)/\Gamma(1-x)$. The kinematic function S of momenta takes the same form as the one from the amplitude in bosonic string (so $S(P_1, P_2, \cdots, P_m)$ just represents the bosonic string counterpart).

For RR sector it is given by

$$A(P_1, P_2, \ldots, P_m) = \mu^{2-m} \left[\prod_{j=1}^{m} \mu^{-iP_i} \gamma\left(\frac{1}{2} + iP_j\right) \right] S(\sqrt{2}P_1, \sqrt{2}P_2, \ldots, \sqrt{2}P_m) \,, \tag{2.5}$$

[2] Due to reflection of the Liouville potential, only one linear combination is allowed; this is consistent with the fact that there is only one type of D-branes (electric), as we argue later.

[3] We suppress an overall delta function $\delta(\sum_i P_i)$ arising from energy conservation.

where we again find the same function S. Only scattering amplitudes involving an even number of RR vertex operators are non-zero. Mixed amplitudes have a similar structure: apart from the leg factors they are similar to the bosonic string theory amplitudes as above.

There is an essential difference between the scattering amplitudes (2.4) and (2.5). From the form of the leg factors in the two equations, we see that the zero momentum NSNS tachyon decouples from the spectrum while the zero momentum RR scalar does not [36]. This fact was explained in [36]. The wavefunction of the zero momentum tachyon is not normalizable, and therefore it decouples from the spectrum. On the other hand, the wavefunction of the zero momentum RR scalar is normalizable and need not decouple.

The results summarized above show that the type 0 theory has essentially a similar perturbative structure as the bosonic string in two dimensions. They imply that the type 0 theory should also admit a dual matrix model description closely related to $c = 1$ matrix model, a point already made in [36].

2.2 Rolling tachyon computation

In two dimensional type 0B string theory, we can consider the decay of unstable D0-branes. The brane decay is described by an exact boundary conformal field theory (BCFT). We consider the case where[4] the rolling tachyon field is represented by the $N = 1$ boundary interaction (half-Sbrane) [23, 25, 26]

$$\mu_B \int_{\partial\Sigma} d\sigma \; \eta\psi e^{\frac{1}{2}X_0} \,, \tag{2.6}$$

where η is the boundary fermion (as usual in BSFT formulation [34]) of unstable D-branes. The D0-branes are described by the $(1,1)$ type degenerate boundary state as argued in [5, 6, 45, 46]. In the $N = 1$ super Liouville theory, the boundary states and one point functions have been computed in [12, 13] and below we use their results. A $(1,1)$ type D0-brane is localized in the strong coupling region, and the open string tachyon living on it has the standard mass $m^2 = -1/2\alpha'$.

First let us consider the emission amplitude (see [17, 5] for bosonic string case) of the NSNS tachyon field T_{cl}. The closed string one point function of the vertex e^{iEX_0} can be computed by analytical continuation as done in [26]

$$A_t = \langle e^{iEX_0} \rangle = -\frac{1}{\sqrt{2\pi}} (\sqrt{2}\mu_B \pi)^{-2iE} \frac{\pi}{\sinh(\pi E)}. \tag{2.7}$$

The Liouville part of the one point function of the vertex $e^{(iP+\frac{Q}{2})\phi}$ on the (1,1) type D-brane has been computed in [12]

$$A_\phi = \left\langle e^{(iP+\frac{Q}{2})\phi} \right\rangle = -i(\mu_0 \gamma(1)\pi)^{-iP} \frac{\Gamma(iP)\sinh(\pi P)}{\Gamma(-iP)} \,. \tag{2.8}$$

[4] It is also possible to consider tachyon fields of the form $\tilde{\lambda}\cosh(x^0/2)$ or $\tilde{\lambda}\sinh(x^0/2)$ with the "Hartle-Hawking" contour defining the analytic continuation [17, 5]. We have to replace λ with $\sin(\pi\tilde{\lambda})^2$ or $\sinh(\pi\tilde{\lambda})^2$ respectively, (see [21]).

Thus the emission rate of on-shell ($E = P$) closed string tachyon field is given by (up to a finite constant)

$$A_{NS} = A_t A_\phi = \mu^{-iP} \frac{\Gamma(iP)}{\Gamma(-iP)} e^{-iP\log\lambda}\,, \tag{2.9}$$

where we renormalized the cosmological constant as $\mu \equiv \mu_0\gamma(1)\pi$ and also defined a parameter of the boundary interaction by $\lambda \equiv 2\pi^2\mu_B^2$.

Next consider the emission amplitude of the RR-field C. In this case due to the spin field of both ψ_0 and ψ_1 we get the factor $\cosh(\pi P)$ instead of $\sinh(\pi P)$ in the timelike and spatial parts of the amplitude. In the final expression both cancel, and we end up with

$$A_R = \mu^{-iP} \frac{\Gamma(\frac{1}{2}+iP)}{\Gamma(\frac{1}{2}-iP)} e^{-iP\log\lambda}\,. \tag{2.10}$$

One important common property of these emission amplitudes is the appearance of momentum dependent phase factors (leg factors)

$$e^{i\delta_{NS}} = \mu^{-iP} \frac{\Gamma(iP)}{\Gamma(-iP)}\,, \tag{2.11}$$

$$e^{i\delta_R} = \mu^{-iP} \frac{\Gamma(\frac{1}{2}+iP)}{\Gamma(\frac{1}{2}-iP)}\,. \tag{2.12}$$

As in the bosonic string [5, 6], this suggest that in a dual matrix model description, these phases may be interpreted as standard leg factors relating matrix model operators and closed string fields.

Note also the appearance of another phase factor $e^{-iP\log\lambda}$. In half S-branes, the parameter λ can be changed by a time translation. Therefore, this phase can be interpreted as a time delay. This phase can be also reproduced in the matrix model description [5].

3. Matrix model description

We would like to propose that type 0B string theory in two dimensions is also described by a one dimensional bosonic matrix model of a hermitean $N \times N$ matrix T. Following the recent proposal of [4]–[6] the matrix T is an open string tachyon field on N unstable $D0$ branes. The ten dimensional type 0 theory contains double spectrum of D-branes: the so called electric branes $|-\rangle$ and magnetic branes $|+\rangle$ [9]–[11]. However, in the two dimensional type 0 theory there is only one kind of D-brane $|-\rangle$ (we call it electric) as can be shown from the general analysis of boundary states in super Liouville theory [12]. This implies that we have no fermionic fields on D-branes. D0 branes in the theory are described by a localized boundary state (corresponding to the $(1,1)$ degenerate state) for the Liouville direction ϕ times a Neumann boundary state for the time direction X^0. The open string spectrum on such branes consists of a tachyon field T and a non-dynamical gauge field A_0. There are no physical open string oscillator modes.

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A similar proposal for type 0A theory can be obtained by considering a system of N charged D0-branes and N anti D0-branes. Even though the tachyon field becomes a complex scalar, the degrees of freedom after projection under the gauge group $U(N)\times U(N)$ are almost the same as in the case of unstable D0-branes in type 0B theory.[5]

The action is naturally assumed to be $S = \beta \int dtL$ with the lagrangian

$$L = \frac{1}{2}\,\mathrm{Tr}(D_t T)^2 - \mathrm{Tr}\, U(T)\,, \tag{3.1}$$

and where

$$D_t T = \partial_t T - i[A_0, T] \tag{3.2}$$

is the covariant derivative. $U(T)$ is the tachyon potential. The gauge field A_0 acts as a Lagrange multiplier that projects onto SU(N) singlet wavefunctions.

The lagrangian above is of the same form as the Matrix model lagrangian used to describe the bosonic string theory with two important differences. First the potential $U(T)$ is symmetric under the Z_2 transformation $T \to -T$, that is $U(T) = U(-T)$ (for example, see the review [47]). Second expanding the potential near the quadratic maximum at $T = 0$, we get

$$U(T) = -\frac{1}{4\alpha'}T^2 + O(T^4)\,, \tag{3.3}$$

that is the mass of the tachyon is given by $m^2 = -1/2\alpha'$ in string units. This differs from the bosonic case by a factor of 1/2. To make contact with the results in the bosonic matrix model (we follow the convention in [1]) we set $\alpha' = 1/2$. To compare with the results in super Liouville theory in which $\alpha' = 2$, we rescale at the end energies and momenta by a factor of 2: $(E, k) \to (2E, 2k)$.

Since the theory has an exact Z_2 symmetry, gauge invariant single trace operators fall naturally into two categories. Operators which are even under $T \to -T$ and operators which are odd. Correlation functions in a state invariant under this Z_2 symmetry of an odd number of odd operators are therefore zero. No such selection rule applies to even operators.

The singlet sector of the Matrix model is exactly solvable due to the fact that the N eigenvalues of the matrix T act as free non-relativistic fermions, and, therefore it can be described by Slater determinant wavefunctions of N variables. The eigenvalues of the matrix T are denoted by λ_i. The hamiltonian is given by

$$\left(\sum_{i=1}^{N} h_i\right)\Psi(\lambda) = E\Psi(\lambda)\,, \tag{3.4}$$

where

$$h_i = -\frac{1}{2\beta^2}\frac{d^2}{d\lambda_i^2} + U(\lambda_i)\,. \tag{3.5}$$

Since the single particle hamiltonian is invariant under the parity transformation $\lambda_i \to -\lambda_i$, the single-particle eigenfunctions have definite parity. They are even or odd under parity.

[5]We thank L. Motl for discussions on this point.

The ground state wavefunction for a single particle bound state will be even. The many particle ground state is obtained by filling[6] the first N levels up to a fermi level $-\mu_F$ *on both sides of the maximum.* The fermi level is measured from the local maximum of the potential. We can choose N even so that the ground state is invariant under the Z_2 symmetry mentioned above. Then the fermi sea is symmetric.

The double scaling limit is obtained by sending $\mu_F \to 0$ and $\beta \to \infty$ keeping $\mu = \beta\mu_F$ fixed. The parameter μ is proportional to $1/g_s$ and has to be kept large in perturbation theory. This limit zooms near the top of the potential near which

$$L = \mathrm{Tr}\left[\frac{1}{2}\left(\frac{dT}{dt}\right)^2 + \frac{1}{2}T^2 + \cdots\right]. \tag{3.6}$$

We will often describe the theory in Euclidean time $x = -it$ and analytically continue.

The two sides of the fermi sea are independent in perturbation theory, that is to all orders in the $g_s \sim 1/\mu$ expansion. The mixing occurs through non-perturbative effects such as single eigenvalue tunneling. Tunneling is suppressed by factors of order $O(\exp(-\mu))$. Thus in the perturbative regime we can focus attention on each side of the fermi sea separately. We call the two sides left(-) and right(+). We essentially have two identical decoupled systems with a parity operator that interchanges them. States in the system can be taken to be linear combinations of

$$|\psi\rangle = a|+\rangle + b|-\rangle\,. \tag{3.7}$$

The parity operator is then the 2×2 Pauli matrix σ_1 with non-zero off-diagonal unit entries. Operators are naturally diagonal in this basis. They act on each side separately without mixing them. Even operators commute with parity. They are proportional to the 2×2 identity matrix. Odd operators anticommute with parity. Therefore, odd operators can be taken to be of the diagonal form

$$\int d\lambda f(\lambda)\rho(\lambda)\sigma_3\,, \tag{3.8}$$

where we have restricted λ positive and $f(\lambda)$ is an odd function of λ. Here $\rho(\lambda)$ is the density of eigenvalues (or fermion bilinear). The ground state can be taken to be invariant under parity. In computing correlation functions in the ground state amounts to taking a trace over the 2×2 diagonal matrices. Then we see that correlators of an odd number of odd operators vanish. All other correlators are identical to the standard case of the bosonic matrix model assuming we take the ground state to be $(|\mu,+> + |\mu,->)/\sqrt{2}$, where the notation indicates that both sides of the potential have been filled up to a fermi level $-\mu$.[7]

The second quantized hamiltonian for a system of free fermions on each side is taken to be (see [1, 29, 30, 49, 52])

$$\hat{H}_\pm = \int_0^\infty dy\left[\frac{1}{2}\frac{\partial\Psi^\dagger}{dy}\frac{\partial\Psi}{dy} - \frac{y^2}{2}\Psi^\dagger\Psi + \mu\left(\Psi^\dagger\Psi - \frac{N}{2}\right)\right], \tag{3.9}$$

[6]Such a kind of model with two fermi seas was already considered before in e.g. [29, 48].

[7]The states $|\mu,\pm>$ are not states of definite parity. They are good ground states in perturbation theory. Non-perturbatively, only the left/right symmetric state is a good ground state.

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where y is the coordinate in the eigenvalue direction. We follow the conventions of [1]. To proceed, it is convenient to bosonize the fermions. To this extent one introduces new chiral fermionic variables $\Psi_L(t,\tau)$ and $\Psi_R(t,\tau)$, where τ is defined through

$$v(y) = \frac{dy}{d\tau} = \sqrt{y^2 - 2\mu}\,, \tag{3.10}$$

the velocity of the classical trajectory of a particle at the fermi level. Then the hamiltonian is free and relativistic in terms of the new chiral fermion fields up to interaction terms of order $1/v^2$ [1]. These interaction terms generate the S-matrix. In the large τ region, v is large, and therefore the relativistic hamiltonian approximates well $\hat{H}$.

The relativistic fermion Ψ_L is bosonized [1, 49, 50, 51, 52, 5] as

$$\Psi_L(\tau+t)|0\rangle = e^{2\sqrt{\pi}X_L(\tau+t)}|0\rangle\,. \tag{3.11}$$

A similar relation holds for the right movers as well. The field X lives on the half line with Dirichlet boundary conditions at $\tau = 0$. It can be expanded as

$$X(x,\tau) = \int dx e^{-iqx} \int dk \sin(k\tau)\tilde{X}(q,k)\,. \tag{3.12}$$

The boundary conditions insure that the fermion current density vanishes at the boundary at $\tau = 0$. Finally, we note that the fermion density bosonizes as follows [1]

$$\Psi_L^\dagger \Psi_L + \Psi_R^\dagger \Psi_R = -\frac{1}{\sqrt{\pi}}\partial_\tau X\,. \tag{3.13}$$

3.1 Matching operators and string states

We begin our analysis by considering a typical macroscopic loop operator in the standard bosonic theory:

$$\int dx e^{iPx}\,\mathrm{Tr}\ e^{-lT(x)}\,. \tag{3.14}$$

In the lorentzian theory this operator gives the following leg factor

$$\mu^{-iP}\frac{\Gamma(2iP)}{\Gamma(-2iP)}\,, \tag{3.15}$$

where we have translated the result for now in our $\alpha' = 2$ notation. Naively this leads to crucially different results from (2.11). In fact, we get poles at $iP = -1/2, -1, -3/2, \ldots$ in eq. (3.15), while in the NSNS sector leg factor poles occur at $iP = -1, -2, \ldots$ and in the RR sector at $iP = -1/2, -3/2, \ldots$. These poles have an important physical meaning in the Euclidean theory. They correspond to resonances in the Euclidean amplitudes due to the presence of extra discrete states [1]. These states are remnants of oscillator modes of the string.

This observation suggests that the space of operators in $c = 1$ bosonic matrix model can be divided naturally into two sectors corresponding to the NSNS and RR sectors of type 0B theory. The operator (3.14) does not have definite parity under $T \leftrightarrow -T$. It is a mixture of both odd and even operators. We propose that even operators in the matrix model describe the NSNS sector and odd ones the RR sector. This is a natural guess since the model we propose has an exact Z_2 symmetry, the symmetry $T \leftrightarrow -T$ of the tachyon field. Indeed the closed string couplings on unstable D-branes respect this symmetry as we will see below.

This choice is also natural from another perspective. By appropriately amputating correlation functions of such operators, we can obtain the scattering amplitudes in the Euclidean theory, as in the bosonic string theory [1]. As we mentioned before correlators of an odd number of odd operators vanish in the symmetric ground state. Scattering amplitudes in the RR sector obey a selection rule: only scattering amplitudes involving an even number of RR vertex operators are non-zero.

To study scattering amplitudes involving the NSNS sector tachyon we propose to study the small l limit of correlators of the following operator[8]

$$O_{NSNS}(q,l) = \int dx e^{iqx}\, \mathrm{Tr}[e^{-lT(x)^2}] \,. \tag{3.16}$$

We will do all computations setting $\alpha' = 1/2$ and rescale to the $\alpha' = 2$ notation in the end. The standard 'puncture' operator $P(q)$ should be the leading term in the small l expansion of eq. (3.16):

$$P(q) \sim l^{-|q|/2} O_{NSNS}(q,l) \,. \tag{3.17}$$

The factor of T^2 in the exponential has an intuitive explanation. It is based on an analogy with boundary string field theory (BSFT) [31]–[33]. In BSFT the coupling of an unstable D-brane to closed strings in the NSNS sector is proportional to the tachyon potential given by $e^{-\int d\sigma T^2}$ for superstrings [34] (and roughly $\sim e^{-\int d\sigma T}$ for the bosonic string). The same argument can be generalized for the type 0A case (constructed from $D0-\bar{D}0$ system). We have to replace $e^{-lT(x)^2}$ with $e^{-lT\overline{T}(x)}$ in (3.16) (— for the BSFT description of brane-antibrane systems see [35]).

Upon bosonization (3.16) becomes

$$O_{NSNS}(q,l) \sim \int dx e^{iqx} \int d\tau e^{-ly^2(\tau)} \partial_\tau X \sim i \int_{-\infty}^{\infty} dk F(k,l) k \tilde{X}(q,k) \,, \tag{3.18}$$

where

$$F(k,l) = \int d\tau e^{-ly^2(\tau)} \cos(k\tau) \,. \tag{3.19}$$

Evaluating $F(k,l)$ with $y(\tau) = \sqrt{2\mu}\cosh(\tau)$, we obtain

$$F(k,l) = \frac{e^{-l\mu}}{2} K_{ik/2}(l\mu) \,, \tag{3.20}$$

where K is a modified Bessel function. In the small l limit this function becomes

$$F(k,l) \to \frac{\pi}{4\sin(\frac{ik\pi}{2})} \left(\left(\frac{l\mu}{2}\right)^{-ik/2} \frac{1}{\Gamma(-ik/2+1)} - cc \right) . \tag{3.21}$$

[8]In this section, we will suppress the 2×2 diagonal matrix accompanying the operators.

In computing m-point Euclidean correlators, each such operator will be connected to the rest of the Feynman graph by the scalar field propagator $1/(q^2+k^2)$. We can now perform the k-integral in (3.18) as in [1]. The contour of intergration is deformed in a special way. For the first term in (3.21), we deform the integration so as to pick up the residue of the propagator pole at $k=i|q|$ while for the second term at the pole $k=-i|q|$ [1]. This procedure gives the amputated on-shell Euclidean amplitude for m X quanta times a factor for each external leg (up to numerical factors)

$$\left(\frac{l\mu}{2}\right)^{|q|/2}\Gamma\left(\frac{-|q|}{2}\right). \tag{3.22}$$

This 'leg factor', reproduces the correct pole structure of scattering amplitudes in the Euclidean string theory due to NSNS discrete states. In choosing the operator $O(q,l)$, we allow freedom to multiply it by a smooth function of $|q|$. Thus we can multiply by a factor of $1/\Gamma(|q|/2)$ (since this is a smooth function). This amounts to smearing the local operator in position space.

Translating into the $\alpha'=2$ notation amounts to rescaling the energies q by a factor of 2. The lorentzian theory is obtained by analytic continuation $|q|\to -iP$. We thus obtain the leg factor

$$\mu^{-iP}\frac{\Gamma(iP)}{\Gamma(-iP)}. \tag{3.23}$$

This agrees with (2.11).

Thus in the small l limit, correlators of m properly normalized NSNS operators take the form

$$\mu^{2-m}\prod_{j=1}^{m}\left(\mu^{-iP_j}\frac{\Gamma(iP_j)}{2\Gamma(-iP_j)}\right)S'(\sqrt{2}P_1,\ldots,\sqrt{2}P_m) \tag{3.24}$$

with $\mu^{2-m}S'$ being the usual bosonic matrix model scattering amplitude. Since there is a well-known agreement between the bosonic string theory and the matrix model, the equivalence with the superstring follows. In particular, S' takes the form[9] [1]

$$S'=\prod_{j=1}^{m}(i\sqrt{2}P_j)S\,, \tag{3.25}$$

where S is the bosonic kinematic part appearing in eq. (2.4). After rescaling the momenta as implied by eq. (3.24), the result agrees exactly with the scattering amplitude (2.4). Notice again that our result implies a rescaling of the bosonic string momenta ($\alpha'=2$) by a factor of $\sqrt{2}$ to get to the superstring. This difference arises due to the difference in mass of the open string tachyon field by the same factor. In a previous section we have seen that apart from leg factors, the kinematic parts of the super Liouville amplitudes are reproduced from the bosonic ones up to the same rescaling of the momenta [36].

For the RR sector we need to choose an odd operator. To reproduce the correct leg factor (2.12), the simplest choice seems to be the operator

$$O_{RR}(q,l)=\int dx e^{iqx}\,\mathrm{Tr}\ Te^{-lT^2}\,. \tag{3.26}$$

[9]In our conventions with $\alpha'=2$, the leg factors in the bosonic theory are given by $\Gamma(i\sqrt{2}P)/\Gamma(-i\sqrt{2}P)$.

This operator (3.26) is consistent with the known RR coupling;[10] the RR field C couples to the open string tachyon as $\int C \wedge dT\ e^{-T^2}$ in BSFT [34, 35]. Roughly, the operator (3.26) should be dual to the field strength of C (see also the discussions below).

After bosonization, we get the following wavefunction $F(k,l)$

$$F(k,l) = \int d\tau y e^{-ly^2} \cos(k\tau). \tag{3.27}$$

Since $y = \sqrt{2\mu}\cosh\tau$

$$\begin{aligned} F(k,l) &= \sqrt{2\mu}e^{-l\mu} \int d\tau \cosh(\tau) e^{-l\mu\cosh(2\tau)} \cos(k\tau) \\ &= \sqrt{\frac{\mu}{2}} e^{-l\mu} \int d\tau e^{-l\mu\cosh(2\tau)} \left(\cos[(-i+k)\tau] + \cos[(i+k)\tau]\right). \end{aligned} \tag{3.28}$$

Evaluating the integral we obtain

$$F(k,l) = \frac{1}{2}\sqrt{\frac{\mu}{2}} e^{-l\mu} \left(K_{(1+ik)/2}(l\mu) + K_{(-1+ik)/2}(l\mu)\right). \tag{3.29}$$

Now consider the small l limit. In this limit, we have

$$K_{(1+ik)/2}(l\mu) \to \frac{\pi}{2\sin((ik+1)\pi/2)} I_{(-ik-1)/2}(l\mu) \tag{3.30}$$

and

$$K_{(-1+ik)/2}(l\mu) \to \frac{-\pi}{2\sin((ik-1)\pi/2)} I_{(ik-1)/2}(l\mu). \tag{3.31}$$

The other terms can be dropped as they are smaller by a factor of l. Finally we obtain

$$F(k,l) = \sqrt{\frac{\mu}{2}} \left(\frac{l\mu}{2}\right)^{-1/2} \left[\frac{\pi}{4\sin((ik+1)\pi/2)} \left(\frac{l\mu}{2}\right)^{-ik/2} \frac{1}{\Gamma[(-ik-1)/2+1]} + cc\right]. \tag{3.32}$$

Now let us perform the momentum k integral. We deform the integration as in [1]. For the first term, we deform the integration to pick the pole $k = i|q|$ from the X-propagator while for the complex conjugate the opposite pole. So after the k integral we get the following leg factor (up to numerical factors)

$$\sqrt{\frac{\mu}{2}} \left(\frac{l\mu}{2}\right)^{(|q|-1)/2} \Gamma\left[\frac{(1-|q|)}{2}\right]. \tag{3.33}$$

The leg factor reproduces the correct pole structure arising from the RR discrete states. As before we are free to multiply by a smooth function of q. Thus we can divide by a factor of $\Gamma[(1+|q|)/2]$. The puncture operator is obtained in the small l limit as follows

$$P(q) \sim l^{(-|q|+1)/2} O_{RR}(q,l). \tag{3.34}$$

[10] In the same way we can discuss operators for the RR sector in the two dimensional type 0A theory. However, we are left with no candidates since now gauge invariant operators must be even under the Z_2 symmetry. This is consistent with the fact that there is no dynamical RR field in the type 0A theory.

We then have to rescale q by a factor of 2, and finally continue to the lorentzian theory. We end up with (in the $\alpha' = 2$ notation) the following leg factor

$$\mu^{-iP}\frac{\Gamma(1/2+iP)}{\Gamma(1/2-iP)} \tag{3.35}$$

in agreement with (2.12).

To obtain the correct scattering amplitudes of RR scalars eq. (2.5), we need to relate the RR vertex operator V_{RR} to the odd operator (3.26) by an additional factor of momentum: $|q|V_{RR} \sim O_{RR}$. This factor arises because the operator (3.26) is directly related to the gauge invariant field strength dC rather than C itself, and reflects the non-decoupling of the zero momentum scalar discussed in section 2. The scattering amplitude (2.5) is reproduced if the leg factors in eq. (3.24) are replaced by the momentum dependent factors

$$\left(\frac{\mu^{-iP_j}}{2iP_j}\right)\frac{\Gamma(1/2+iP_j)}{\Gamma(1/2-iP_j)}\,. \tag{3.36}$$

The considerations of this section imply that the perturbative structure of the S-matrix is universal in two dimensional string theory, bosonic and fermionic. Even though quite significant in the Euclidean theory (since they contain physical information), the leg factor phases appearing in the lorentzian S-matrix amplitudes do not affect scattering cross-sections. In fact, they can be removed by a unitary transformation on the states of the theory. Non-perturbatively, we should expect a different picture to emerge. In the bosonic case, non-perturbative effects such as eigenvalue tunneling can spoil unitarity of the S-matrix and the stability of the vacuum. One typically expects the open string tachyon potential to be unbounded from below. In the fermionic case however, worldsheet supersymmetry implies that the potential is bounded from below, and the closed string vacuum is stable. We do not expect non-perturbative violations of unitarity of the S-matrix in this case.

3.2 Rolling eigenvalues and brane decay

Our considerations above suggest that we can identify decaying D0-branes in type 0B theory with rolling eigenvalues in the matrix model as in [5, 4]. A decaying brane results in a coherent state of closed strings [17]. Similarly a rolling eigenvalue down the potential results in a coherent state for the bosonic field X of the form (3.11) [5]. The relation of this field to the string theory scalar fields is non-local. Our considerations in the previous section suggest that in momentum space the relation is just a momentum dependent phase.

In the case we consider, we can obtain coherent states on both sides of the fermi sea by considering rolling of eigenvalue pairs. Since the NSNS sector is described by operators even under parity $T \leftrightarrow -T$, one particle states of the NSNS tachyon correspond to symmetric one-particle states of X of the form

$$|k\rangle_{NSNS} = e^{i\delta(k)_{NS}}\frac{(|k,+\rangle + |-k,-\rangle)}{\sqrt{2}}\,. \tag{3.37}$$

Such a disturbance of the fermi sea is left/right symmetric. Similarly for the RR scalar we get

$$|k\rangle_{RR} = e^{i\delta(k)_R} \frac{(|k,+\rangle - |-k,-\rangle)}{\sqrt{2}} \,. \tag{3.38}$$

Since the RR sector is described by odd operators, the expectation value of these in the symmetric ground state and an odd state of the form (3.38) is non-zero. The phases are the leg factors eq. (2.11). The time delay in the matrix model is reproduced by considering classical trajectories of eigenvalues that start away from the local maximum [5].

3.3 Macroscopic operators and open string partition function

Before finishing this paper, let us consider an interesting relation between the annulus amplitude of Euclidean unstable D0-branes and the two point function of macroscopic loop operators of the form (3.16). Such D-objects are defined by Neumann boundary conditions along the Liouville direction, and Dirichlet conditions for the Euclidean time direction X^0. From the point of view of the lorentzian theory, these objects are D-instantons. In a recent paper [15], it is proposed that such branes in the bosonic string are described by macroscopic loop operators of the form (1.1). Here we would like to check this correspondence for our two dimensional fermionic string case. The amplitude describes NSNS closed string exchange.

After integration of the closed string channel modulus, the annulus partition function is given by

$$Z = \int dP \int dE \frac{e^{iP(x-x')} \cos(2\pi P\sigma)\cos(2\pi P\sigma')}{\sinh^2(\pi E)(E^2+P^2)} \,, \tag{3.39}$$

where the parameter σ is related to the boundary cosmological constant by $\mu_B^2 = \frac{2\mu_0 \sinh^2(\pi\sigma)}{\cos(\pi b^2/2)}$ [12].

Define the Laplace transform of the operator[11] (3.16) by

$$W(\sigma,q) = \int \frac{dl}{l} e^{-\mu_B^2 l} O_{NSNS}(k,l) \,. \tag{3.40}$$

Though we need a renormalization of both μ_B and μ, we can cancel it by scaling l. Then by using the formula

$$\int_0^\infty \frac{dl}{l} e^{-\mu l \cosh(2\pi\sigma)} K_{ik}(\mu l) = \frac{\pi\cos(2\pi\sigma k)}{k\sinh(\pi k)} \,, \tag{3.41}$$

we can see that the two macroscopic loop correlator $\int dq e^{iq(x-x')} \langle W(\sigma,q) W(\sigma,-q)\rangle$ is equal to the annulus amplitude (3.39). Note that the boundary cosmological constant leads to the term $\mu_B^2 \int_{\partial\Sigma} d\sigma e^{\phi}$ in the superstring case. The two-point function can be obtained using eq. (3.18) and (3.20). It would be interesting to obtain such a correspondence for the RR operators as well.

[11] Here we consider $|+\rangle$ branes. For $|-\rangle$ branes, we need to put an extra factor $e^{l\mu}$ in the definition of dual operator.

4. Conclusions

In this paper we explored holographic dual matrix models of two dimensional type 0 theory. We mainly focused our analysis on the seemingly simpler case of type 0B theory. The tachyon field on (infinitely) many unstable D-branes plays the role of the elementary matrix field in the matrix model. In the fermionic string the tachyon potential becomes left-right symmetric and the theory admits a Z_2 symmetry. We checked our proposal for the duality by comparing tree level scattering amplitudes of closed strings with their matrix theory counterparts. The dual matrix model is the same as that of the bosonic string in the double scaling limit. However, the exact Z_2 symmetry of the fermionic model hints at the existence of a stable vacuum even at the non-perturbative level. It would be an exciting future direction to compute non-perturbative quantities in the symmetric matrix model and examine their implications for the fermionic string. Finally, it would be interesting to obtain a precise description of type 0A in terms of a matrix model of a complex field.

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